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Almost periodic solutions of difference equations with discrete argument on metric space

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ALMOST PERIODIC SOLUTIONS OF DIFFERENCE EQUATIONS WITH DISCRETE ARGUMENT ON METRIC SPACE

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Abstract. We obtain conditions for existence of almost periodic solutions of difference equations with discrete argument on metric space.

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1. INTRODUCTION

Let M be a metric space with metric ρ_M. Fix an arbitrary element $a \in M$. Denote by $\mathfrak{M}$ the metric space of sequences $\mathbf{x} = (\mathbf{x}(n))_{n\in\mathbb{Z}}$ for which

$$\sup_{n\in\mathbb{Z}} \rho_M(\mathbf{x}(n), a) < \infty$$

with metric

$$\rho_{\mathfrak{M}}(\mathbf{x}_1, \mathbf{x}_2) = \sup_{n\in\mathbb{Z}} \rho_M(\mathbf{x}_1(n), \mathbf{x}_2(n)).$$

Define the shift operator $S_m : \mathfrak{M} \to \mathfrak{M}$, $m \in \mathbb{Z}$, by the formulae

$$(S_m\mathbf{x})(n) = \mathbf{x}(n+m), \ \ n \in \mathbb{Z}.$$

Definition 1. The sequence $\mathbf{x} \in \mathfrak{M}$ is called almost periodic if the set $\overline{\{S_m\mathbf{x} : m \in \mathbb{Z}\}}$ is compact in $\mathfrak{M}$.

Denote by $\mathfrak{B}$ the metric space of sequences $\mathbf{x} \in \mathfrak{M}$ which are almost periodic.

Let $\mathcal{K}$ be the set of compact sets K of the metric space M and let $R(\mathbf{x})$ be the set $\{\mathbf{x}(n) : n \in \mathbb{Z}\}$. Fix an arbitrary compact set $K \subset \mathcal{K}$. We denote by $\mathfrak{D}_K$ the set of all elements of $\mathbf{x} \in \mathfrak{M}$ for each of which $R(\mathbf{x}) \subset K$.

Definition 2. The operator $\mathbf{H} : \mathfrak{M} \to \mathfrak{M}$ is called almost periodic if for every set $K \in \mathcal{K}$ and a sequence $(m_k)_{k\geqslant 1}$ of whole numbers there exists a subsequence $(m_{k_l})_{l\geqslant 1}$, which

$$\lim_{l_1\to\infty,\ l_2\to\infty} \sup_{\mathbf{x}\in\mathfrak{D}_K} \rho_{\mathfrak{M}}\left(S_{m_{l_1}}\mathbf{H}S_{-m_{l_1}}\mathbf{x}, S_{m_{l_2}}\mathbf{H}S_{-m_{l_2}}\mathbf{x}\right) = 0.$$

Note that the almost periodic operator $\mathbf{H} : \mathfrak{M} \to \mathfrak{M}$ can not be almost periodic by Bochner [2, 3]. However, the almost periodic by Bochner operator $\mathbf{H} : \mathfrak{M} \to \mathfrak{M}$ be almost periodic by Definition 2.

Let Λ be a bounded subset of the space M. Define the diameter $\operatorname{diam}\Lambda$ of the set Λ by the equality

$$\operatorname{diam}\Lambda = \sup\{\rho_M(x, y) : x, y \in \Lambda\}.$$

Consider the almost periodic difference operator $\mathbf{F} : \mathfrak{M} \to \mathfrak{M}$ defined by the formulae

$$(\mathbf{F}\mathbf{x})(n) = F(n, \mathbf{x}(n), \mathbf{x}(n + m_1), \ldots, \mathbf{x}(n + m_k)), \quad n \in \mathbb{Z},$$

where $\mathbf{x} \in \mathfrak{M}$, $k \in \mathbb{N}$, $m_1, \ldots, m_k \in \mathbb{Z}$ and $F : \mathbb{Z} \times M^{k+1} \to M$ is operator such that

$$\operatorname{diam} F(\mathbb{Z} \times M_1 \times \ldots \times M_{k+1}) < +\infty$$

for all bounded sets $M_1, \ldots, M_{k+1}$.

Consider the difference equation

$$\mathbf{F}\mathbf{x} = \mathbf{h}, \tag{1.1}$$

where $\mathbf{h} \in \mathfrak{B}$.

The aim of this work is to find conditions under which the bounded solutions of equation (1.1) are almost periodic.

In the study of equation (1.1) will use a functional defined on the set of solutions of an equation of the sets of values which are subsets of compact sets.

2. Functional δ

Fix an arbitrary set $K \in \mathcal{K}$. Let $N(\mathbf{F}, K)$ be the set of all solutions of equation (1.1), each of which $R(\mathbf{x}) \subset \mathcal{K}$ and $\overline{R(\mathbf{x})} \neq K$. Suppose that $N(\mathbf{F}, K) \neq \varnothing$.

Fix an arbitrary element $\mathbf{x}^* \in N(\mathbf{F}, K)$. Let

$$r(\mathbf{x}^*, K) = \sup\left\{\rho_M(x, y) : x \in \overline{R(\mathbf{x}^*)}, y \in K\right\}.$$

Due to the inequality $N(\mathbf{F}, K) \neq \varnothing$

$$r(\mathbf{x}^*, K) > 0.$$

Also fix the arbitrary number $\varepsilon \in [0, r(\mathbf{x}^*, K)]$. We denote by $\Omega(\mathbf{x}^*, K, \varepsilon)$ the set of all elements of $\mathbf{y} \in \mathfrak{M}$, each of which

$$R(\mathbf{y}) \subset K$$

and

$$\rho_{\mathfrak{M}}(\mathbf{y}, \mathbf{x}^*) \geq \varepsilon.$$

Consider the functional

$$\delta(\mathbf{x}^*, K, \varepsilon) = \inf_{\mathbf{y} \in \Omega(\mathbf{x}^*, K, \varepsilon)} \rho_{\mathfrak{M}}(\mathbf{F}\mathbf{y}, \mathbf{F}\mathbf{x}^*). \tag{2.1}$$

First analogous functionals have been proposed by the author in the papers [7,8,10] for the study of nonlinear almost periodic equations

$$x(t+1) = f(t, x(t)), \;\; t \in \mathbb{R},$$
$$\frac{dx(t)}{dt} = f(t, x(t)), \;\; t \in \mathbb{R},$$

and

$$f(t, x(t)) = 0, \;\; t \in \mathbb{R},$$

with continuous operator $f : \mathbb{R} \times E \to E$. Here E is a Banach space. Analogous functional for nonlinear difference equation

$$x(n+1) = g(n, x(n)), \;\; n \in \mathbb{Z},$$

used in [9].

3. Main Result

We give conditions for the existence of almost periodic solutions of equation (1.1), in contrast to the well-known theorem Amerio of almost periodic solutions of nonlinear differential equations (see [1,4]) do not use the H-class of equation (1.1). In the case of linear differential equations using H-classes of these equations is essential [5,6].

The main result of this paper reads as follows

Theorem 1. *Let us suppose that* $K \in \mathcal{K}$, $\mathbf{z} \in N(\mathbf{F}, K)$, $\operatorname{diam} R(\mathbf{z}) \neq 0$ *and*

$$\delta(\mathbf{z}, K, \varepsilon) > 0 \tag{3.1}$$

for each $\varepsilon \in (0, r(\mathbf{z}, K))$. *Then solution* $\mathbf{z}$ *of equation* (1.1) *is almost periodic.*

Proof. Assume that the solution $\mathbf{z} \in N(\mathbf{F}, K)$ of the equation (1.1) is not element of the space $\mathfrak{B}$. Then there exists a sequence $(S_{m_p}\mathbf{z})_{p\geq 1}$ such that each subsequence $(S_{k_p}\mathbf{z})_{p\geq 1}$ is divergent. Consequently, for some numbers $p_r \in \mathbb{N}$, $q_r \in \mathbb{N}$, $r \geq 1$, and $\gamma \in (0, \operatorname{diam} R(\mathbf{z}))$

$$\rho_{\mathfrak{M}}(S_{k_{p_r}}\mathbf{z}, S_{k_{q_r}}\mathbf{z}) \geq \gamma, \;\; r \geq 1.$$

Because

$$\rho_{\mathfrak{M}}(\mathbf{z}, S_{-k_{p_r}} S_{k_{q_r}}\mathbf{z}) \geq \gamma, \;\; r \geq 1,$$

and therefore

$$S_{-k_{p_r}} S_{k_{q_r}}\mathbf{z} \in \Omega(\mathbf{z}, K, \gamma), \;\; r \geq 1. \tag{3.2}$$

Based on the inclusion of $\mathbf{h} \in \mathfrak{B}$, without loss of generality of the proof, we can assume that

$$\lim_{r\to\infty} \rho_{\mathfrak{M}}\left(S_{-k_{p_r}}\mathbf{h}, S_{-k_{q_r}}\mathbf{h}\right) = 0. \tag{3.3}$$

Note that diam $R(\mathbf{z})) \leq r(\mathbf{z}, K))$. Without loss of generality we can assume that the sequence $(S_{k_p}\mathbf{F}S_{-k_p}\mathbf{x})_{p\geq 1}$ converges uniformly on $\mathfrak{D}_K$. Then

$$\lim_{p,q\to\infty} \sup_{\mathbf{x}\in\mathfrak{D}_K} \rho_{\mathfrak{M}}(S_{k_p}\mathbf{F}S_{-k_p}\mathbf{x}, S_{k_q}\mathbf{F}S_{-k_q}\mathbf{x}) = 0. \tag{3.4}$$

We show that

$$\delta(\mathbf{z}, K, \gamma) = 0. \tag{3.5}$$

It is obvious that by (2.1) and (3.4)

$$\delta(\mathbf{z}, K, \gamma) = \inf_{\mathbf{y}\in\Omega(\mathbf{z},K,\gamma)} \rho_{\mathfrak{M}}(\mathbf{Fy}, \mathbf{Fz}) \leq \rho_{\mathfrak{M}}(\mathbf{F}S_{-k_{p_r}}S_{k_{q_r}}\mathbf{z}, \mathbf{Fz}), \quad r \geq 1. \tag{3.6}$$

Note that

$$\begin{aligned}
&\rho_{\mathfrak{M}}(\mathbf{F}S_{-k_{p_r}}S_{k_{q_r}}\mathbf{z}, \mathbf{Fz}) = \\
&= \rho_{\mathfrak{M}}\left(S_{-k_{p_r}}\left(S_{k_{p_r}}\mathbf{F}S_{-k_{p_r}}\right)S_{k_{q_r}}\mathbf{z}, S_{-k_{q_r}}\left(S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}\right)S_{k_{q_r}}\mathbf{z}\right) \\
&\leq \rho_{\mathfrak{M}}\left(S_{-k_{p_r}}\left(S_{k_{p_r}}\mathbf{F}S_{-k_{p_r}}\right)S_{k_{q_r}}\mathbf{z}, S_{-k_{p_r}}\left(S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}\right)S_{k_{q_r}}\mathbf{z}\right) \\
&+ \rho_{\mathfrak{M}}\left(S_{-k_{p_r}}\left(S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}\right)S_{k_{q_r}}\mathbf{z}, S_{-k_{q_r}}\left(S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}\right)S_{k_{q_r}}\mathbf{z}\right) \\
&= \rho_{\mathfrak{M}}\left(\left(S_{k_{p_r}}\mathbf{F}S_{-k_{p_r}}\right)S_{k_{q_r}}\mathbf{z}, \left(S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}\right)S_{k_{q_r}}\mathbf{z}\right) + \rho_{\mathfrak{M}}\left(S_{-k_{p_r}}S_{k_{q_r}}\mathbf{h}, S_{-k_{q_r}}S_{k_{q_r}}\mathbf{h}\right) \\
&\leq \sup_{\mathbf{x}\in\mathfrak{D}_K} \rho_{\mathfrak{M}}\left(S_{k_{p_r}}\mathbf{F}S_{-k_{p_r}}\mathbf{x}, S_{k_{q_r}}\mathbf{F}S_{-k_{q_r}}\mathbf{x}\right) + \rho_{\mathfrak{M}}\left(S_{-k_{p_r}}\mathbf{h}, S_{-k_{q_r}}\mathbf{h}\right), \quad r \geq 1.
\end{aligned}$$

Therefore, based on (3.3), (3.4) and (3.6) the equality (3.5) is true.

This relation contradicts (3.1).

Thus, the assumption that the solution $\mathbf{z} \in N(\mathbf{F}, K)$ of the equation (1.1) is not element of the space $\mathfrak{B}$, is false.

So, the proof is complete. □

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